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The Blume–Emery–Griffiths model at an infinitely many ground states interface and exponential decay of correlations at all non-zero temperatures

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Abstract
In this paper we study the spin–spin correlation function decay properties of the Blume–Emery–Griffiths (BEG) model with Hamiltonian located on the interface between the disordered and the anti-quadrupolar phases. On this interface, the BEG model has infinitely many ground state configurations. We show that, for any dimension d, there exists a parameter value, y_d, below which the spin–spin correlation function with zero boundary condition decays exponentially fast at all non-zero temperatures. This result suggests that reentrant behaviour predicted by mean-field and numerical calculations may be absent for those values of parameters.

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1. Introduction

The Blume–Emery–Griffiths (BEG) model was introduced in the 1970s in order to explain some of the physical properties of He^3 and He^4 mixtures [1] and since then it has attracted a lot of attention and has been used in other applications such as ternary fluids [2, 3] and Langmuir mono-layers [4, 5]. Its Hamiltonian is given by

$$H(\sigma) = -J\sum_{(i,j)}\sigma_i\sigma_j - K\sum_{(i,j)}\sigma_i^2\sigma_j^2 + D\sum_{(i)}\sigma_i^2$$

where $i \in \mathbb{Z}^d$ is a lattice site, (i, j) is a nearest-neighbour bond, σ_i is the spin variable at the lattice site i taking values $0, +1, -1$. The parameters J, K, D satisfy $J > 0, -\infty < K, D < +\infty$. Defining $y = K/J$ and $x = -D/(2dJ)$, we rewrite the Hamiltonian as follows:

$$H(\sigma) = -J\sum_{(i,j)}\left[\sigma_i\sigma_j + y\sigma_i^2\sigma_j^2 + x\left(\sigma_i^2 + \sigma_j^2\right)\right] \equiv \sum_{(i,j)} H_{i,j}(\sigma). \qquad (1)$$

Without loss of generality, we will assume $J = 1$. The phase diagram in the xy-plane at $T = 0$ is obtained by minimizing the spin pair energy $H_{ij}(\sigma)$ and it consists of three distinct regions:

- $F = \{x, y \in \mathbb{R} : 1 + y + 2x > 0 \text{ and } 1 + y + x > 0\}$;
- $D = \{x, y \in \mathbb{R} : 1 + y + 2x < 0 \text{ and } x < 0\}$;
- $A = \{x, y \in \mathbb{R} : 1 + y + x < 0 \text{ and } x > 0\}$.

F is the ferromagnetic region, D is the disordered region and A is the anti-quadrupolar region. In A, as well as on its boundaries A–D and F–D, the model has infinitely many ground state configurations. In A, one can use the polymer expansion to obtain the low-temperature Gibbs measures [6] associated with two distinct equivalence classes of ground states. New features may show up on the phase boundaries. For instance, for the two-dimensional model in the interface A–D, mean-field calculations indicate the presence of reentrance although this is not confirmed by numerical renormalization group calculations (see [7, 8] and references therein). One possible way to decide whether or not there is reentrant behaviour is to look at the decay of the spin–spin correlation function. This point of view was developed in [9] and their theorem 1 gives sufficient conditions for the exponential decay in the two-dimensional model, for all non-zero temperatures.

Following the strategy adopted in [9], in this paper we study the spin–spin correlation function of the d-dimensional BEG model at the interface $x = 0$ and $y < -1$. Specifically, we provide a value y_d of y, given explicitly as a function of d, below which there is exponential decay of correlations for all non-zero temperatures. More specifically, our goal in this paper is to prove the following result:

Theorem 1.1. *Consider the d-dimensional BEG model located on the A–D interface. If β and y are such that*

$$4d^2\, e^{\beta(1+y)}(1 - e^{-\beta}) < 1 \tag{2}$$

then the 0-boundary condition spin–spin correlation function decays exponentially fast:

$$\langle \sigma_i \sigma_j \rangle^0 \leqslant C(\beta, y)\, e^{-m(\beta,y)\|i-j\|} \tag{3}$$

where $C(\beta, y) > 0$ and

$$e^{-m(\beta,y)} = 4d^2\, e^{\beta(1+y)}(1 - e^{-\beta}). \tag{4}$$

Although the BEG model has been studied by many authors using different techniques, as far as we know, there are no rigorous results for the model on the disordered–antiquadrupolar interface which holds for any space dimension. As a consequence of the above theorem, we will show below that, at any dimension d, there is a value y_d of y so that the spin–spin correlation function decays exponentially fast for any $y < y_d$ and any $\beta \geqslant 0$. This result indicates the absence of reentrant behaviour for $y < y_d$.

Corollary 1.1. *Consider the d-dimensional BEG model located on the A–D interface. There exists a negative value y_d such that the 0-boundary condition spin–spin correlation function decays exponentially fast for any $y < y_d$ and for any $\beta \geqslant 0$.*

Proof. Assume theorem 1.1 and let $f(\beta, y) = e^{\beta(1+y)}(1 - e^{-\beta})$. Then, for each $y < -1$ fixed, the maximum of f is attained at $\beta^* = \ln \frac{y}{1+y}$. The function $h(y) = f(\beta^*(y), y)$ is clearly a monotone decreasing function of $|y|$. Hence, if we define y_d as the solution of $h(y) = \frac{1}{4d^2}$, then $4d^2 f(\beta, y) < 1$ for all $y < y_d$ and all $\beta \geqslant 0$, i.e., inequality (2) is satisfied, leading to the exponential decay of the spin–spin correlation function at all non-zero temperatures, as long as $y < y_d$. □

This paper is organized as follows: in section 2 we give some definitions and prove two Griffiths-type inequalities which will be used in section 3 to prove theorem 1.1. We point out that even though its proof is based on a high-temperature approach, the results apply to the whole range of temperatures. In section 4 we make some concluding remarks. For completeness, in the appendix we give a 'low-temperature' version of theorem 1.1. It leads to a temperature range already covered by the results of theorem 1.1.

2. Preliminaries

Given any finite subset $\Lambda \subset Z^d$, we define Λ^* as the set of all unordered nearest-neighbour bonds (i, j) with i and j in Λ. By the boundary of $\Lambda, \partial\Lambda$, we mean the set of all points $i \in \mathbb{Z}^d - \Lambda$, such that $\|i - j\| = 1$ for some j in Λ, where, for any $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, $\|i\| \equiv \sum_{k=1}^d |i_k|$. By Ω_Λ we mean the set of all spin configurations $\{-1, 0, 1\}^\Lambda$. In what follows, we will take $x = 0$ in the Hamiltonian given by (1).

Given any $\Lambda \subset \mathbb{Z}^d$ and any $s \in \Omega_{\partial\Lambda}$ we define the Hamiltonian with s boundary condition

$$H_{\Lambda,s}(\sigma) = \sum_{(i,j)\in\Lambda^*} H_{i,j}(\sigma) + \sum_{(i,j):i\in\Lambda, j\in\partial\Lambda, \|i-j\|=1} \left(\sigma_i s_j + y\sigma_i^2 s_j^2\right). \tag{5}$$

The finite volume expectation is

$$\langle . \rangle_{\Lambda,s} = \frac{\sum_{\sigma\in\Omega_\Lambda} \mathrm{e}^{-\beta H_{\Lambda,s}(\sigma)}}{Z_{\Lambda,s}} \tag{6}$$

where $Z_{\Lambda,s}$ is the partition function:

$$Z_{\Lambda,s} = \sum_{\sigma\in\Omega_\Lambda} \mathrm{e}^{-\beta H_{\Lambda,s}(\sigma)}. \tag{7}$$

Remark. If we drop the second sum in (5) we get the free boundary condition. In the interface we are considering, the energies of the spin pairs 00 and $0\pm$ are zero. Therefore, the '0' and the 'free' boundary conditions are equivalent. For any $A \subset \mathbb{Z}^d$, by $\langle . \rangle_A$ and Z_A we mean $\langle . \rangle_{A,0}$ and Z_A^0, respectively.

We now obtain a graph representation for the Gibbs factor $\mathrm{e}^{-\beta H_\Lambda}$. We start by finding a representation for the Gibbs factor of a spin pair:

$$\begin{aligned}\mathrm{e}^{\beta(\sigma_i\sigma_j + y\sigma_i^2\sigma_j^2)} &= \left(1 + \sigma_i\sigma_j \sinh\beta + \sigma_i^2\sigma_j^2(\cosh\beta - 1)\right)\left(1 + (\mathrm{e}^{\beta y} - 1)\sigma_i^2\sigma_j^2\right) \\ &= 1 + a\sigma_i\sigma_j + b\sigma_i^2\sigma_j^2 \equiv 1 + \rho_{ij}(\sigma)\end{aligned}$$

where

$$\rho_{ij} = a\sigma_i\sigma_j + b\sigma_i^2\sigma_j^2 \tag{8}$$

with

$$a = \mathrm{e}^{\beta y} \sinh\beta \qquad \text{and} \qquad b = \mathrm{e}^{\beta y}(\cosh\beta - 1). \tag{9}$$

Let $\rho_{ij}(\sigma)$ be defined by (8), where a and b are given by (9). Then

$$\begin{aligned}\mathrm{e}^{-\beta H_\Lambda(\sigma)} &= \prod_{(i,k)\in\Lambda^*} \mathrm{e}^{-\beta H_{i,j}(\sigma)} = \prod_{(i,k)\in\Lambda^*} (1 + \rho_{ij}(\sigma)) \\ &= 1 + \sum_{k\geqslant 1} \sum_{(i_i,j_1),\ldots,(i_k,j_k)\in\Lambda^*} \rho_{i_1,j_1}(\sigma)\ldots\rho_{i_k,j_k}(\sigma)\end{aligned} \tag{10}$$

where the second sum is over all un-repeated k-uples $(i_1, j_1)\cdots(i_k, j_k)$. With each term $\rho_{i_1,j_1}\ldots\rho_{i_k,j_k}$ in (10) we associate a geometric object, Γ, which we call a graph. This graph consists of points of Λ and nearest-neighbour bonds of Λ^*, connecting them. The set of points of Γ will be just the union of the points i_l and j_l, $l = 1, \dots, k$ and the set of bonds will be just the union of $(i_1, j_1), \dots, (i_k, j_k)$, which we will denote by Γ^*. To represent geometrically the graph Γ, first we mark with a dot each one of the distinct points of Γ and then we draw a line connecting two points m and n if (m, n) is one of the bonds of Γ^*. In what follows, $|\Gamma^*|$ will denote the cardinality of Γ^*, which we also call the size of Γ, and $|\Gamma|$ will denote the number of points in Γ. Therefore, if we associate 1 with the graph with no bonds, we have the following graph representation for (10):

$$\mathrm{e}^{-\beta H_\Lambda(\sigma)} = \sum_{\Gamma} \prod_{(i_l, j_l)\in\Gamma^*} \rho_{i_l, j_l}(\sigma) \tag{11}$$

where the sum is over all above-defined graphs Γ in Λ.

Given $A \subset \Lambda$ and $\{n_i \in \mathbb{N} : i \in A\}$, define $\sigma^A = \prod_{\{i\in A\}} \sigma_i^{n_i}$. Griffiths' first inequality holds in this case. Here we state this inequality whose proof can be found in [9], proposition $A1$.

Proposition 2.1. *For all $\beta \geqslant 0$, $y \in \mathbb{R}$ and $\Lambda \subset \mathbb{Z}^d$ we have*

$$\langle \sigma^A \rangle_\Lambda \geqslant 0. \tag{12}$$

Now we fix a bond (i, j) in Λ^* and replace the parameters a and b, both appearing in the definition of ρ_{ij}, given by (8), by a_{ij} and b_{ij} respectively. Regarding Z_Λ as a function of a_{ij} and b_{ij}, we have:

Proposition 2.2. *The finite volume partition function Z_Λ is a non-decreasing function of $a_{ij} \geqslant 0$ and $b_{ij} \geqslant 0$.*

Proof. Using the graph representation (11) to rewrite the partition function as a function of a and b, we take derivatives to obtain

$$\frac{\partial Z_\Lambda}{\partial a_{ij}} = Z_{\overline{\Lambda}} \langle \sigma_i \sigma_j \rangle_{\overline{\Lambda}} \tag{13}$$

$$\frac{\partial Z_\Lambda}{\partial b_{ij}} = Z_{\overline{\Lambda}} \left\langle \sigma_i^2 \sigma_j^2 \right\rangle_{\overline{\Lambda}} \tag{14}$$

where $\overline{\Lambda} = \Lambda - \{i, j\}$. Using the positivity of correlations given by proposition 2.1, we get the result. □

3. Proof of theorem 1.1

We first state a result which will be used below and whose proof can be found, for instance, in [10], proposition V.7A.3, p 465.

Proposition 3.1. *The number N_n of connected subgraphs Γ of size $|\Gamma^*| = n$ that pass through a fixed vertex i_0 is no bigger than $(4d^2)^n$.*

Theorem 1.1 is proved by using a high-temperature type argument along the lines developed in [11]. We note that, for the two-point function $\langle \sigma_i \sigma_j \rangle_\Lambda$, only those graphs Γ having a connected component Γ_1 containing i and j will give a non-zero contribution to the numerator of $\langle \sigma_i \sigma_j \rangle_\Lambda$. For such graphs Γ, we have the following decomposition $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $Z_{\Lambda-\Gamma_1}$ be the partition function for the region $\Lambda - \Gamma_1$, obtained from Λ by suppressing the lattice points contained in Γ_1, namely,

$$\begin{aligned} Z_{\Lambda-\Gamma_1} &= \sum_{\{\sigma\}\in\Omega_{\Lambda-\Gamma_1}} \mathrm{e}^{-\beta\sum_{(i,j)\in(\Lambda-\Gamma_1)^*} H_{ij}(\sigma)} \\ &= \sum_{\Gamma_2\subset(\Lambda-\Gamma_1)} \sum_{\{\sigma\}\in\Omega_{\Lambda-\Gamma_1}} \prod_{(i,j)\in\Gamma_2^*} \rho_{ij}. \end{aligned}$$

From the graph representation (11) and from the relation above, we get that the spin–spin correlation $\langle\sigma_i\sigma_j\rangle_\Lambda$ can be written as

$$\langle\sigma_i\sigma_j\rangle_\Lambda = Z_\Lambda^{-1} \sum_{\Gamma_1\supset\{i,j\}} \left(\sum_{\{\sigma\}\subset\Omega_{\Gamma_1}} \sigma_i\sigma_j \prod_{(k,l)\in\Gamma_1^*} \rho_{k,l}(\sigma) \right) Z_{\Lambda-\Gamma_1}. \tag{15}$$

To proceed, we observe that

$$0 \leqslant \frac{Z_{\Lambda-\Gamma_1}}{Z_\Lambda} \leqslant 3^{-|\Gamma_1|} \tag{16}$$

which follows from proposition 2.2 applied to all a and b associated with bonds (k, l) for which either k or l belongs to Γ_1. From (15) and (16), we have

$$|\langle\sigma_i\sigma_j\rangle_\Lambda| \leqslant \sum_{\Gamma_1\supset\{i,j\}} \left(\sum_{\{\sigma\}\subset\Omega_{\Gamma_1}} |\sigma_i\sigma_j| \prod_{(k,l)\in\Gamma_1^*} |\rho_{k,l}(\sigma)| \right) 3^{-|\Gamma_1|}. \tag{17}$$

The minimal size of a graph Γ_1 containing i and j is given by $\|i - j\|$. Since $|\sigma_i| \leqslant 1$ and $|\rho_{i,j}(\sigma)| \leqslant a + b$, it follows from (17) that

$$\begin{aligned} |\langle\sigma_i\sigma_j\rangle_\Lambda| &\leqslant \sum_{\Gamma_1\supset\{i,j\}} \left(\sum_{\{\sigma\}\subset\Omega_{\Gamma_1}} |\sigma_i\sigma_j| \prod_{(k,l)\in\Gamma_1^*} |\rho_{k,l}(\sigma)| \right) 3^{-|\Gamma_1|} \\ &\leqslant \sum_{n\geqslant\|i-j\|} \sum_{\Gamma_1\supset\{i,j\},|\Gamma_1^*|=n} \left(\sum_{\{\sigma\}\subset\Omega_{\Gamma_1}} |\sigma_i\sigma_j| \prod_{(k,l)\in\Gamma_1^*} |\rho_{k,l}(\sigma)| \right) 3^{-|\Gamma_1|} \\ &\leqslant \sum_{n\geqslant\|i-j\|} \left(\sum_{\Gamma_1\supset\{i,j\},|\Gamma_1^*|=n} 1 \right) (a+b)^n \leqslant \sum_{n\geqslant\|i-j\|} N_n (a+b)^n \end{aligned}$$

where N_n is the number of finite connected graphs of size n passing through i. From proposition 3.1, this number can be bounded from above by $(4d^2)^n$. Using this bound and the definition of a and b (see (9)), we get

$$|\langle\sigma_i\sigma_j\rangle_\Lambda| \leqslant \sum_{n\geqslant\|i-j\|} (4d^2\,\mathrm{e}^{\beta(1+y)}(1 - \mathrm{e}^{-\beta}))^n.$$

Since the above bound is uniform in Λ, the bound (3) holds true if the condition (2) is satisfied. The pre-factor $C(\beta, y)$ (see equation (3)) is equal to

$$\sum_{n\geqslant 0}(4d^2\,\mathrm{e}^{\beta(1+y)}(1 - \mathrm{e}^{-\beta}))^n.$$

4. Concluding remarks

We have shown that, for any dimension d, there exists a value y_d of y below which the spin–spin correlation function of the BEG model located at the disordered–antiquadrupolar interface with zero boundary condition decays exponentially fast at all non-zero temperatures. In [8] (see figure 5*a*, p 485), comparative (Monte Carlo simulation, renormalization group and mean-field approximations) temperature T versus crystalline field x phase diagrams for the two-dimensional BEG model are shown, for values of y equal to -1 and -2. While mean-field calculations suggest reentrant behaviour, Monte Carlo and RG computations go in the opposite direction. When restricted to two dimensions, our results give $y_2 \simeq -6.395$, see corollary 1.1, indicating that there is no such behaviour for $y < -6.395$. The results of [9], which hold only for two dimensions, give $y_2 \simeq -2.187$. Although their upper bound for y is better than ours, we point out that our approach is technically simpler than theirs and holds for any dimension d.

Although it is clear, we remark that the value of y_d in corollary 1.1 gets closer to -1 if sharper upper bound estimates for N_n are given (see proposition 3.1). Finally, it is interesting to know whether or not the model, located on the A–D interface, has a unique Gibbs measure for all non-zero temperatures.

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Appendix

For completeness, in this appendix we obtain a result similar to theorem 1.1 by performing a low-temperature expansion along the lines of section 3. We recall that the energy of spin pairs is $H_{ij}(00) = H_{ij}(0+) = H_{ij}(0-) = 0$, $H_{ij}(++) = H_{ij}(--) = -(y+1)$, $H_{ij}(+-) = -(y-1)$. Given a configuration σ in Ω_Λ, we define the following subset of Λ^*:

$$\{(k,l) : H_{kl}(\sigma) \neq 0\}. \tag{A.1}$$

With the set given by (A.1) we associate a geometric object, Γ, which we call a graph. This graph consists of all bonds of the set (A.1) and their end points. By construction, the spin pairs in Γ are either $++$, $--$ or $+-$. Also, there is no interaction between spin variables inside Γ and those outside because, by construction, 0 is the boundary condition on the boundary of Γ. By the spin flipping symmetry of the Hamiltonian, only those configurations σ whose associated graph Γ has a connected component, Γ_1, containing i and j will contribute for the numerator of the spin–spin correlation function. This allows us to factorize any configuration σ_Λ, which contributes to the numerator of the spin–spin correlation, as $(\sigma_{\Lambda-\Gamma_1}, \sigma_{\Gamma 1})$, where $\sigma_{\Lambda-\Gamma_1} \in \Omega_{\Lambda-\Gamma_1}$ and $\sigma_{\Gamma_1} \in \{-1,1\}^{\Gamma_1}$, for some connected graph, Γ_1, containing both i and j. There is no interaction between the spins in $\sigma_{\Lambda-\Gamma_1}$ and the spins in σ_{Γ_1}. This observation is the basis of our low-temperature expansion.

Theorem A.1. *Consider the d-dimensional BEG model located on the $A - D$ interface. If β and y are such that*

$$4ed^2\, \mathrm{e}^{(1+y)\beta} < 1 \tag{A.2}$$

then the 0-boundary condition spin–spin correlation function decays exponentially fast:

$$\langle \sigma_i \sigma_j \rangle^0 \leqslant \mathrm{e}^{-m(\beta,y)\|i-j\|}$$

where $C(\beta, y) > 0$ *and*

$$\mathrm{e}^{-m(\beta,y)} = 4ed^2\,\mathrm{e}^{(1+y)\beta}.$$

Proof. There is a graph representation to the Gibbs factor

$$\mathrm{e}^{-\beta H_\Lambda(\sigma)} = \prod_{(i,j)\in\Gamma^*} \mathrm{e}^{-\beta H_{ij}} \equiv \prod_{(i,j)\in\Gamma^*} \rho_{ij}(\sigma)$$

which naturally leads to the following representation for spin–spin correlation:

$$\langle\sigma_i\sigma_j\rangle_\Lambda = Z_\Lambda^{-1} \sum_{\Gamma_1\supset\{i,j\}} \left(\sum_{\sigma\in\{-1,1\}^{\Gamma_1}} \sigma_i\sigma_j \prod_{(k,l)\in\Gamma^*} \rho_{k,l}(\sigma) \right) Z_{\Lambda-\Gamma_1}. \tag{A.3}$$

Note that for fixed Γ_1, we have

$$Z_\Lambda \geqslant \sum_{\{\sigma\}}^{*} \mathrm{e}^{-\beta H(\sigma)} = Z_{\Lambda-\Gamma_1} 2^{\frac{|\Gamma|}{2}} \tag{A.4}$$

where in the sum $\sum^*$ we restrict ourselves to those configurations for which σ_{Γ_1} is a typical anti-quadrupolar ground state, namely, each spin pair is of the form ± 0. From equations (A.3) and (A.4), since $|\rho_{k,l}(\sigma)| \leqslant \mathrm{e}^{\beta(1+y)}$, $|\sigma_i\sigma_j| \leqslant 1$ and $\sum_{\sigma\in\{-1,1\}^{\Gamma_1}} 1 = 2^{|\Gamma_1|} \leqslant 2^{2|\Gamma_1^*|}$, then for any Λ, we have

$$\begin{aligned} |\langle\sigma_i\sigma_j\rangle_\Lambda| &\leqslant \sum_{\Gamma_1\supset\{i,j\}} \mathrm{e}^{(1+y)\beta|\Gamma_1^*|}\,\mathrm{e}^{|\Gamma_1{}^*|} \\ &\leqslant \sum_{n\geqslant\|i-j\|} N_n\,\mathrm{e}^{n((1+y)\beta+1)} \\ &\leqslant \sum_{n\geqslant\|i-j\|} (4ed^2\,\mathrm{e}^{(1+y)\beta})^n. \end{aligned}$$

□

Remark. Observe that condition (2) is stronger than condition (A.2) in the sense that if the latter is satisfied then the former is also satisfied.

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